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Greedy Algorithm for General Biorthogonal Systems

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We consider biorthogonal systems in quasi-Banach spaces such that the greedy algorithm converges for each $x \in X$ (quasi-greedy systems). We construct quasi-greedy conditional bases in a wide range of Banach spaces. We also compare the greedy algorithm for the multidimensional Haar system with the optimal m-term approximation for this system. This substantiates a conjecture by Temlyakov.

Key Words: quasi-greedy basis; conditional basis; biorthogonal system; Haar system.

1. INTRODUCTION

We consider a general quasi-Banach space X with the norm $\|\cdot\|$ such that for all $x, y \in X$ we have $\|x+y\| \leqslant \alpha(\|x\|+\|y\|)$. The letter α will *always* in this paper denote this constant. It is well known (cf. [3] Lemma 1.1.) that in such a situation there is a p, $0<p\leqslant 1$, such that $\|\sum_n x_n\| \leqslant 4^{1/p}(\sum_n \|x_n\|^p)^{1/p}$. Recall that a biorthogonal system in a quasi-Banach space X is a family $(x_n, x_n^*)_{n\in F} \subset X \times X^*$ such that $x_n^*(x_m)$ equals zero whenever $n \neq m$ and equals one if $n=m$. Here F is any countable index set. We fix a biorthogonal system $(x_n, x_n^*)_{n\in F}$ in X such that $\operatorname{span}(x_n)_{n\in F}=X$ and $\inf_{n\in F}\|x_n\|>0$ and $\sup_{n\in F}\|x_n^*\|<\infty$. This implies that for each $x\in X$ we have $\lim_{n\to\infty} x_n^*(x)=0$. For each $x\in X$ and $m=1, 2, \ldots$ we define

$$\mathscr{G}_m(x)=\sum_{n\in A} x_n^*(x)\, x_n, \tag{1}$$

where $A\subset F$ is a set of cardinality m such that $|x_n^*(x)| \geqslant |x_k^*(x)|$ whenever $n\in A$ and $k \notin A$. The above set may not be uniquely defined but if this happens we take any such set. The operator $\mathscr{G}_m(x)$ is a non-linear and

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discontinuous operator. We will use linear projection operators P_A defined for any finite subset $A \subset F$ by the formula $P_A(x) = \sum_{n \in A} x_n^*(x)\, x_n$.

This simple theoretical algorithm is a model for a procedure which is widely used in numerical applications. It also raises many interesting questions in functional analysis. The reader can find in [9]–[11] a more detailed description of the connections with purely numerical questions and results for some concrete systems.

In this paper we will use standard Banach space notation as explained in detail in [12] or [6]. The basic reference for simple facts we are going to use about quasi-Banach space is [3].

Definition 1. A biorthogonal system $(x_n, x_n^*)_{n \in F}$ is called a quasi-greedy system if for each $x \in X$ the sequence $\mathcal{G}_m(x)$ converges to x in norm. If this system is a basis we will use the phrase quasi-greedy basis.

Clearly every unconditional basis is a quasi-greedy basis. Let us recall the definition of the best m-term approximation. For $x \in X$ and $m = 0, 1, ...$ we put

$$\sigma_m(x) = \inf \left\{ \left\| x - \sum_{n \in A} a_n x_n \right\| : |A| \leqslant m \text{ and } a_n\text{'s are scalars} \right\}. \tag{2}$$

Definition 2. A basis $(x_n, x_n^*)_{n \in F}$ is called greedy if there exists a constant C such that for every $x \in X$ we have $\|x - \mathcal{G}_m(x)\| \leqslant C\sigma_m(x)$.

After the research reported in this paper was practically completed I received the preprint [4] where the above terminology was introduced, so I decided to follow this terminology in this note. It is shown in [4] that each greedy basis is unconditional and an example of a conditional quasi-greedy basis is given.

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2. QUASI-GREEDY SYSTEMS

Let us start with some general results. The following theorem gives some natural equivalent conditions for quasi-greedy systems.

Theorem 1. *The following conditions are equivalent*:

1. *The system $(x_n, x_n^*)_{n \in F}$ is quasi-greedy.*
2. *For each $x \in X$ the series $\sum_{n=1}^{\infty} x_{\sigma(n)}^*(x)\, x_{\sigma(n)}$ converges to x where σ is an ordering of F such that $(|x_{\sigma(n)}^*(x)|)_{n=1}^{\infty}$ is a decreasing sequence.*

3. *There exists a constant* C *such that for any* $x \in X$ *and* $m = 1, 2, ...$ *we have* $\|\mathcal{G}_m(x)\| \leqslant C\,\|x\|$.

This theorem is basically a uniform boundedness result. However, since the operator $\mathcal{G}_m$ is non-linear and discontinuous we have to give a direct proof.

Proof. Clearly $1 \Leftrightarrow 2$

$3 \Rightarrow 1$. Since the convergence is clear for x's with finite expansion in the biorthogonal system, let us assume that x has an infinite expansion. Take $x_0 = \sum_{n \in A} a_n x_n$ such that $\|x - x_0\| < \varepsilon$ where A is a finite set and $a_n \neq 0$ for $n \in A$. If we take m big enough we can ensure that $\mathcal{G}_m(x - x_0) = \sum_{n \in B} x_n^*(x - x_0)\, x_n$ with $B \supset A$ and $\mathcal{G}_m(x) = \sum_{n \in B} x_n^*(x)\, x_n$. Then

$$\begin{aligned}\|x - \mathcal{G}_m(x)\| &\leqslant \alpha(\|x - x_0\| + \|x_0 - \mathcal{G}_m(x)\|)\\ &\leqslant \alpha(\varepsilon + \|\mathcal{G}_m(x_0 - x)\|) \leqslant \alpha(C+1)\,\varepsilon.\end{aligned}$$

This gives 1.

$1 \Rightarrow 3$. Let us start with the following lemma.

LEMMA 1. *If* 3 *does not hold, then for each constant* K *and each finite set* $A \subset F$ *there exists a finite set* $B \subset F$ *disjoint from* A *and a vector* $x = \sum_{n \in B} a_n x_n$ *such that* $\|x\| = 1$ *and* $\|\mathcal{G}_m(x)\| \geqslant K$ *for some* m.

Proof. Let us fix M to be the maximum of the norms of the (linear) projections $P_\Omega(x) = \sum_{n \in \Omega} x_n^*(x)\, x_n$ where $\Omega \subset A$. Let us start with a vector x^1 such that $\|x^1\| = 1$ and $\|\mathcal{G}_m(x^1)\| \geqslant K_1$ where K_1 is a big constant to be specified later. Without loss of generality we can assume that all numbers $|x_n^*(x^1)|$ are different. For $x^2 = x^1 - \sum_{n \in A} x_n^*(x^1)\, x_n$ we have $\|x^2\| \leqslant \alpha(M+1)$ and $\mathcal{G}_m(x^1) = \mathcal{G}_k(x^2) + P_\Omega(x^1)$ for some $k \leqslant m$ and $\Omega \subset A$. Thus $\|\mathcal{G}_k(x^2)\| \geqslant \frac{K_1}{\alpha} - M$ and for $x^3 = x^2 \cdot \|x^2\|^{-1}$ we have $\|\mathcal{G}_k(x^3)\| \geqslant (K_1/\alpha - M)/(1+M)\,\alpha$. Let us put

$$\delta = \inf\{\,|x_n^*(\mathcal{G}_k(x^3))| : x_n^*(\mathcal{G}_k(x^3)) \neq 0\}$$

and take a finite set B_1 such that for $n \notin B_1$ we have $|x_n^*(x^3)| \leqslant \delta/2$. Let us take η very small with respect to $|B_1|$ and $|A|$ and find x^4 with finite expansion such that $\|x^3 - x^4\| < \eta$. If η is small enough we can modify all coefficients of x^4 from B_1 and A so that the resulting x^5 will have its k biggest coefficients the same as x^3 and $\|x^4 - x^5\| < \delta$. Moreover x^5 will have the form $x^5 = \sum_{n \in B} x_n^*(x^5)\, x_n$ with B finite and disjoint from A. Since $\|x^5\| \leqslant \alpha(\|x^4\| + \delta) \leqslant \alpha(\alpha(1+\eta) + \delta) \leqslant C\alpha$ and $\mathcal{G}_k(x^5) = \mathcal{G}_k(x^3)$, for $x = x^5 \cdot \|x^5\|^{-1}$ we get $\|\mathcal{G}_k(x)\| \geqslant (\frac{K_1}{\alpha} - M)(\alpha + \alpha M)^{-1}\,(C\alpha)^{-1}$ which can be made $\geqslant K$ if we take K_1 big enough. ∎

Using Lemma 1 we can apply the standard gliding hump argument to get a sequence of vectors $y_n = \sum_{k \in B_n} a_k x_k$ with sets B_n disjoint and $\|y_n\| = 1$, a decreasing sequence of positive numbers $\varepsilon_n \leqslant 2^{-n}$ such that if $x_k^*(y_n) \neq 0$ then $|x_k^*(y_n)| \geqslant \varepsilon_n$ and a sequence of integers m_n such that $\|\mathcal{G}_{m_n}(y_n)\| \geqslant 2^n \prod_{j=1}^{n-1} \varepsilon_j^{-1}$. Now we put $x = \sum_{n=1}^{\infty} (\prod_{j=1}^{n-1} \varepsilon_j)\, y_n$. This series is clearly convergent in X. If we write $x \sim \sum_{n \in F} b_n x_n$ we infer that

$$\inf \left\{ |b_n| : n \in \bigcup_{s=1}^{j} B_s \text{ and } b_n \neq 0 \right\} \geqslant \prod_{s=1}^{j} \varepsilon_s \geqslant \max \left\{ |b_n| : n \notin \bigcup_{s=1}^{j} B_s \right\}.$$

This implies that for $k = \sum_{s=1}^{j-1} |B_s| + m_j$ we have

$$\mathcal{G}_k(x) = \sum_{n \leqslant j} \left(\prod_{s=1}^{n-1} \varepsilon_s \right) y_n + \mathcal{G}_{m_j} \left(\prod_{s=1}^{j} \varepsilon_s \right) y_{j+1}$$

so $\|G_k(x)\| \geqslant \alpha^{-1} (\prod_{s=1}^{n-1} \varepsilon_s)\, \|\mathcal{G}_{m_j} y_{j+1}\| - C \geqslant 2^{j+1}/\alpha - C$. Thus $\mathcal{G}_m(x)$ does not converge to x. ∎

Let us now introduce the following definition:

Definition 3. A system $(x_n, x_n^*)_{n \in F}$ is called unconditional for constant coefficients if there exist constants C and $c > 0$ such that for each finite $A \subset F$ and each sequence of signs $(\varepsilon_n)_{n \in A} = \pm 1$ we have

$$c \left\| \sum_{n \in A} x_n \right\| \leqslant \left\| \sum_{n \in A} \varepsilon_n x_n \right\| \leqslant C \left\| \sum_{n \in A} x_n \right\|. \tag{3}$$

Definition 3 is justified by the following observation.

Proposition 2. *Every quasi-greedy system is unconditional for constant coefficients.*

Proof. For a given sequence of signs $(\varepsilon_n)_{n \in A}$ let us define the set $A_1 = \{n \in A : \varepsilon_n = 1\}$. For each $\delta > 0$ and $\delta < 1$ we apply Theorem 1 and we get

$$\left\| \sum_{n \in A_1} x_n \right\| \leqslant C \left\| \sum_{n \in A_1} x_n + \sum_{n \in A \setminus A_1} (1 - \delta)\, x_n \right\|.$$

Since this is true for each $\delta > 0$ we easily obtain the right hand side inequality in (3). The other inequality follows by analogous arguments. ∎

Remark. Let us clarify a bit the problem of non-uniqueness of $\mathcal{G}_m(x)$. In our definition of quasi-greedy system we require that for each $x \in X$ we can choose (if there is a choice) a $\mathcal{G}_m(x)$ such that $\mathcal{G}_m(x) \to x$. The statements 2 and 3 of Theorem 1 are also to be understood in this way—in 2 we think about *one* convergent decreasing rearrangement and in 3 we think about *one* good $\mathcal{G}_m(x)$. However Proposition 2 immediately imply that those reservations are not essential. It shows that if $(x_n, x_n^*)_{n\in F}$ is quasi-greedy, then *any* series $\sum_{n=1}^{\infty} x_{\sigma(n)}^*(x)\, x_{\sigma(n)}$ such that $(|x_{\sigma(n)}^*(x)|)$ is decreasing, converges to x. This implies that we have convergence for any choice of $\mathcal{G}_m(x)$.

Our definitions of a quasi-greedy system and of the operator $\mathcal{G}_m$ depend on the normalisation of the system considered. This, however, is not essential. Namely we have

PROPOSITION 3. *Suppose that $(x_n, x_n^*)_{n\in F}$ is a quasi-greedy system as discussed. Let $(\lambda_n)_{n\in F}$ be a sequence of numbers such that $0<a=: \inf_{n\in F}|\lambda_n| \leqslant b =: \sup_{n\in F}|\lambda_n|<\infty$. Then the system $(\lambda_n x_n, x_n^*/\lambda_n)_{n\in F}$ is also quasi-greedy.*

Proof. By homogeneity we can and will assume that $b=1$. Let $\mathcal{G}^1_m$ be the greedy approximation operator corresponding to the system $(\lambda_n x_n, x_n^*/\lambda_n)_{n\in F}$. Let us fix $x\in X$ and a natural number m. Explicitly we have $\mathcal{G}^1_m(x)=\sum_{n\in A} x_n^*(x)\, x_n$ where $A\subset F$ is a set of cardinality m such that $|x_n^*(x)/\lambda_n| \geqslant |x_s^*(x)/\lambda_s|$ whenever $n\in A$ and $s\notin A$. Let us write $\eta=\inf_{n\in A} |x_n^*(x)|$ and let $V=\{n\in F: |x_n^*(x)\geqslant \eta\}$ and $U=\{n\in F: |x_n^*(x)|\geqslant \eta/a\}$. We put $|V|=k$ and $|U|=l$. Clearly $U\subset V$ so $l\leqslant k$. Using those notations we can write

$$\begin{aligned}\mathcal{G}^1_m(x) &= \mathcal{G}_k(x) - \sum_{s\in B} x_s^*(x)\, x_s \\ &= \mathcal{G}_l(x) + \left(\mathcal{G}_k(x) - \mathcal{G}_l(x) - \sum_{s\in B} x_s^*(x)\, x_s\right),\end{aligned} \tag{4}$$

where B is a certain subset of $V\setminus U$ and so we know that for $s\in B$ we have

$$\eta \leqslant |x_s^*(x)| \leqslant \eta/a. \tag{5}$$

Clearly $\mathcal{G}_k(x)-\mathcal{G}_l(x)=\sum_{n\in V\setminus U} x_n^*(x)\, x_n$. Note that for each finite set $D\subset F$ and each set of numbers $(a_n)_{n\in D}$ we have

$$\left\|\sum_{n\in D} a_n x_n\right\| \leqslant C \left\|\sum_{n\in D} x_n\right\|. \tag{6}$$

To see (6) we write a dyadic expansion of each a_n namely $a_n = \pm\sum_{s=1}^{\infty} a(n,s)\,2^{-s}$ where $a(n,s)=0,1$. Then from Proposition 2 we have

$$\left\| \sum_{n\in D} a_n x_n \right\| = \left\| \sum_{s=1}^{\infty} 2^{-s} \sum_{n\in D} \pm a(n,s)\, x_n \right\|$$

$$\leqslant 4^{1/p} \sum_{s=1}^{\infty} 2^{-sp} \left\| \sum_{n\in D} \pm a(n,s)\, x_n \right\| \leqslant C.$$

Thus we infer from (5) and (6) that

$$\begin{aligned}\left\| \mathscr{G}_k(x) - \mathscr{G}_l(x) - \sum_{s\in B} x_n^*(x)\, x_n \right\| &\leqslant C \left\| \sum_{n\in V\setminus U} (\eta/a)\, x_n \right\| \\ &\leqslant C^2/a \left\| \sum_{n\in V\setminus U} x_n^*(x)\, x_n \right\| \\ &\leqslant C^2/a\, \|\mathscr{G}_k(x) - \mathscr{G}_l(x)\|. \end{aligned} \tag{7}$$

Comparing (4) and (7) we infer that $\|\mathscr{G}_m^1(x)\| \leqslant C'\,\|x\|$ so by Theorem 1 the system $(\lambda_n x_n, x_n^*/\lambda_n)_{n\in F}$ is a quasi-greedy system. ∎

Remark. Proposition 2 allows us to show that the trigonometric system in $L_p(\mathbb{T})$ with $1 \leqslant p \leqslant \infty$ is quasi-greedy only if $p=2$, because only then it is unconditional for constant coefficients. To see this observe that for $1 < p \leqslant \infty$ we have $\|\sum_{n=1}^{N} e^{int}\|_p^p \sim N^{p-1}$ and for $p=1$ we have $\|\sum_{n=1}^{N} e^{int}\| \sim \log(N+1)$. On the other hand for $1 \leqslant p < \infty$ the average over all signs $\pm$ of $\|\sum_{n=1}^{N} \pm e^{int}\|_p^p$ can be written as $\int_0^1 \|\sum_{n=1}^{N} r_n(s)\, e^{int}\|_p^p\, ds$ where $r_n(s)$ are classical Rademacher functions. It is well known in the theory of type and cotype of Banach spaces (see [12]) and easily follows from the Khintchine's inequality that $\int_0^1 \|\sum_{n=1}^{N} r_n(s)\, e^{int}\|_p^p\, ds \sim N^{p/2}$ so the trigonometric system in $L_p(\mathbb{T})$ $(1 \leqslant p < \infty)$ an be quasi-greedy only when $p-1 = p/2$ i.e. when $p=2$. In the case $p=\infty$ we can invoke the Rudin–Shapiro polynomials i.e. polynomials $\phi_N = \sum_{n=1}^{N} \pm e^{int}$ such that $\|\phi_N\|_\infty \leqslant C\sqrt{N}$. This argument reproves results from [11] remark 2.

Now we will discuss examples of conditional quasi-greedy bases. Let us recall

Definition 4. A biorthogonal system (resp. basis) $(x_n, x_n^*)_{n\in F}$ is a p-Besselian system, $0 < p < \infty$ if there exists a constant C such that for each $x \in X$ we have

$$\left(\sum_{n\in F} |x_n^*(x)|^p \right)^{1/p} \leqslant C\, \|x\|.$$

A 2-Besselian system (resp. basis) will be called Besselian.

THEOREM 2. *Suppose X is a quasi-Banach space with Besselian basis $(x_n, x_n^*)_{n=1}^{\infty}$. The space $X \oplus \ell_2$ has a quasi-greedy basis. If the basis $(x_n, x_n^*)_{n=1}^{\infty}$ is conditional we get a conditional quasi-greedy basis in $X \oplus \ell_2$.*

Before we start the proof let us recall some classical notions from Banach space theory. If X and Y are Banach spaces then the symbol $X \oplus Y$ denotes the direct sum of those spaces i.e. the space of all pairs (x, y) with $x \in X$ and $y \in Y$. This is a linear space with coordinatewise addition and scalar multiplication. As a norm on $X \oplus Y$ we can take $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$. We will identify an element $x \in X$ with a pair $(x, 0) \in X \oplus Y$, so in particular $x + y$ means (x, y) whenever $x \in X$ and $y \in Y$. If we have a sequence of Banach spaces $(X_n)_{n=1}^{\infty}$ and a number $1 \leqslant p < \infty$ then $(\sum_{n=1}^{\infty} X_n)_p$ denotes the space of all sequences $(x_n)_{n=1}^{\infty}$ such that $x_n \in X_n$ for $n = 1, 2, ...$ and $\|(x_n)\| = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p} < \infty$.

Proof. Let us recall some facts about Olevskii matrices (cf. [7]). For $k = 1, 2, ...$ we define $2^k \times 2^k$ matrices $A^k = (a_{ij}^{(k)})_{i,j=1}^{2^k}$ by the following formulas

$$a_{i1}^{(k)} = 2^{-k/2} \qquad \text{for} \quad i = 1, 2, ..., 2^k$$

and for $j = 2^s + v$ with $1 \leqslant v \leqslant 2^s$ and $s = 0, 1, 2, ..., k-1$ we put

$$a_{ij}^{(k)} = \begin{cases} 2^{(s-k)/2} & \text{for} \quad (v-1)\, 2^{k-s} < i \leqslant (2v-1)\, 2^{k-s-1} \\ -2^{(s-k)/2} & \text{for} \quad (2v-1)\, 2^{k-s-1} < i \leqslant v 2^{k-s} \\ 0 & \text{otherwise.} \end{cases}$$

One easily checks that the A^k are orthonormal matrices and there exists a constant C_p such that for all i, k we have

$$\sum_j |a_{ij}^{(k)}|^p \leqslant C_p \qquad \text{for} \quad p > 0. \tag{8}$$

Note that A^k is a matrix which maps an orthonormal Haar-like system in $\mathbb{R}^{2^k}$ onto the unit vector basis. We put $N_k = 2^{10^k}$ and define S_k so that $S_1 = N_1 - 1$ and $S_{k+1} - S_k = N_k - 1$. Let $(e_r)_{r=1}^{\infty}$ denote the unit vector basis in ℓ_2. Let us denote by $(g_s)_{s=1}^{\infty} \subset X \oplus \ell_2$ the following basis

$$x_1, e_1, ..., e_{S_1}, x_2, e_{S_1+1}, ..., e_{S_2}, x_3, e_{S_2+1}, ..., e_{S_3}, x_4, ...$$

To each block $\{x_k, e_{S_{k-1}+1}, ..., e_{S_k}\}$ we apply the matrix A^{10^k} to get a new system

$$\psi_i^k = \frac{x_k}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a_{ij}^{10^k} e_{S_{k-1}+j}. \tag{9}$$

The system $\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ...,$ ordered in this fashion will be denoted by $(\psi_j)_{j=1}^\infty$. It is clear that $0 < \inf_j \|\psi_j\| \leqslant \sup_j \|\psi_j\| < \infty$ and that $(\psi_j)_{j=1}^\infty$ is a complete biorthogonal system in $X \oplus \ell_2$ with the biorthogonal functionals given by the formula

$$\psi_i^{k\,*} = \frac{x_n^*}{\sqrt{N_k}} + \sum_{j=2}^{N_k} a_{ij}^{10^k} e_{S_{k-1}+j}^*.$$

It is also a basis in $X \oplus \ell_2$.

Since the system $(g_j)_{j=1}^\infty$ is a basis it suffices to check that for each k the system $(\psi_i^k)_{i=1}^{N_k}$ have uniformly bounded basis constant. But on each subspace $\operatorname{span}\{x_k, e_{S_{k-1}+1}, ..., e_{S_k}\}$ the $\ell_2^{N_k}$ norm and the norm in $X \oplus \ell_2$ are uniformly equivalent, so any orthonormal basis in this finite dimensional space has uniformly bounded basis constant in $X \oplus \ell_2$. If $(x_n)_{n=1}^\infty$ is a conditional basis then $(\psi_j)_{j=1}^\infty$ is also conditional because $(x_n)_{n=1}^\infty$ is a block basis of $(\psi_j)_{j=1}^\infty$.

Thus we still have to show:

If $f = \sum_{j=1}^\infty a_j \psi_j \in X \oplus \ell_2$ with $\|f\| = 1$, and σ is a permutation such that $|a_{\sigma(j)}|$ is a decreasing sequence, then the series $\sum_{j=1}^\infty a_{\sigma(j)} \psi_{\sigma(j)}$ converges in $X \oplus \ell_2$.

First observe that $(g_s)_{s=1}^\infty$ is a Besselian basis in $X \oplus \ell_2$, so we can define an operator $I: X \oplus \ell_2 \to \ell_2$ as $I(\sum_{s=1}^\infty a_s g_s) = (a_s)_{s=1}^\infty$. Since $(\psi_j)_{j=1}^\infty$ is obtained from $(g_s)_{s=1}^\infty$ by the action of a unitary matrix we infer that $(\psi_j)_{j=1}^\infty$ is also Besselian and $(I\psi_j)_{j=1}^\infty$ is an orthonormal basis in ℓ_2. Let P denote the natural projection from $X \oplus \ell_2$ onto ℓ_2. Note that $I \mid \{0\} \oplus \ell_2$ is an isometric embedding. Let Q denote the orthogonal projection onto $I(\{0\} \oplus \ell_2)$.

Let us write f as a double sum $f = \sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k \psi_i^k$. Since $(\psi_j)_{j=1}^\infty$ is Besselian we get

$$\sum_{k=1}^\infty \sum_{i=1}^{N_k} |b_i^k|^2 \leqslant C. \tag{10}$$

This implies that the series $\sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k I(\psi_i^k)$ converges in ℓ_2 in any order so also $\sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k QI(\psi_i^k)$ converges in any order. This implies that also the series $\sum_{k=1}^\infty \sum_{i=1}^{N_k} b_i^k P\psi_i^k$ converges in $X \oplus \ell_2$ in any order.

Thus we have to study the convergence of the series $\sum_{k=1}^{\infty}\sum_{i=1}^{N_k} b_i^k(I-P)\,\psi_i^k=\sum_{k=1}^{\infty}\sum_{i=1}^{N_k} b_i^k(x_k/\sqrt{N_k})$ but ordered in such a way that coefficients $|b^k{}_i|$ form a decreasing sequence. Let us denote

$$\Lambda_k=\{i: N_k^{-1}<|b_i^k|<N_k^{-1/10}\}$$

$$\Lambda_k'=\{i: |b_i^k|\leqslant N_k^{-1}\}$$

$$\Lambda_k''=\{i: |b_i^k|\geqslant N_k^{-1/10}\}.$$

Note that for each k we have $\sum_{i\in\Lambda_k'}|b_i^k|/\sqrt{N_k}\leqslant N_k\cdot 1/N_k\cdot 1/\sqrt{N_k}\leqslant N_k^{-1/2}$. Thus the series $\sum_{k=1}^{\infty}\sum_{i\in\Lambda_k'}(b_i^k/\sqrt{N_k})\,x_k$ is absolutely convergent. It follows from (10) that for each k, $\sum_{i=1}^{N_k}|b_i^k|^2\leqslant C$ so $C\geqslant\sum_{i\in\Lambda_k''}|b_i^k|^2\geqslant|\Lambda_k''|\,N_k^{-1/5}$ which gives $|\Lambda_k''|\leqslant CN_k^{1/5}$. From this we get

$$\sum_{k=1}^{\infty}\sum_{i\in\Lambda_k''}\frac{|b_i^k|}{\sqrt{N_k}}\leqslant\sum_{k=1}^{\infty}|\Lambda_k''|\frac{1}{\sqrt{N_k}}\leqslant\sum_{k=1}^{\infty}N_k^{-3/10}<\infty$$

so the series $\sum_{k=1}^{\infty}\sum_{i\in\Lambda_k''}(b_i^k/\sqrt{N_k})\,x_k$ is absolutely convergent. Since $N_{k+1}^{-1/10}\leqslant N_k^{-1}$ we see that the decreasing permutation of the series $\sum_{k=1}^{\infty}\sum_{i\in\Lambda_k}b_i^k\psi_i^k$ has to take place inside each Λ_k. But in this case (since (ψ_k) is a basis in $X\oplus\ell_2$) the series over k converges in $X\oplus\ell_2$. From the Schwarz inequality and (10) we get

$$\sum_{i\in\Lambda_k}\frac{|b_i^k|}{\sqrt{N_k}}\leqslant\left(\sum_{i\in\Lambda_k}|b_i^k|^2\right)^{1/2}(|\Lambda_k|\,N_k^{-1})^{1/2}=o(1)$$

so we see that the series $\sum_{k=1}^{\infty}\sum_{i\in\Lambda_k}(b^k/\sqrt{N_k})\,x_k$ converges when rearranged in decreasing order of the coefficients (b_i^k). ∎

This proof is a modification of an argument used in the main result in [5].

Let us note some corollaries from the above construction.

COROLLARY 4. *A separable, infinite dimensional Hilbert space has a quasi-greedy conditional basis.*

Proof. It is known (cf. e.g. [7]) that a Hilbert space has a conditional Besselian basis, so writing $\ell_2=\ell_2\oplus\ell_2$ we get the claim. ∎

COROLLARY 5. *The space ℓ_p for $1<p<\infty$ has a conditional quasi-greedy basis.*

Proof. It is well known (cf. [8]) that ℓ_p is isomorphic to $(\sum_{n=1}^{\infty}\ell_2^n)_p$. Let us fix $(\psi_j)_{j=1}^{\infty}$, a conditional quasi-greedy basis in ℓ_2 which exists by

Corollary 4. Then ℓ_2^n is isometric to $\mathrm{span}(\psi_j)_{j=1}^n$ so the obvious basis in $(\sum_{n=1}^\infty \mathrm{span}(\psi_j)_{j=1}^n)_p$ is a conditional quasi-greedy basis in ℓ_p. ∎

COROLLARY 6. *If X has an unconditional basis and contains a complemented subspace isomorphic to ℓ_p with $1<p<\infty$ then X has a conditional quasi-greedy basis.*

Proof. We write $X \sim Y \oplus \ell_p \sim Y \oplus \ell_p \oplus \ell_p \sim X \oplus \ell_p$ so taking the unconditional basis in the first summand and a conditional quasi-greedy basis in ℓ_p (cf. Corollary 5) we get a conditional quasi-greedy basis in X. ∎

Note that the above Corollary 6 gives the existence of conditional quasi-greedy bases in $L_p[0,1]$ with $1<p<\infty$ and also in H_1.

The following theorem shows that quasi-greedy bases in a Hilbert space are rather close to unconditional bases.

THEOREM 3. *Let $(x_n)_{n=1}^\infty$ be a normalized quasi-greedy basis in a Hilbert space H. Then there exist constants $0<c\leqslant C<\infty$ such that for each $x=\sum_{n=1}^\infty a_n x_n$ we have*

$$c\, \|(a_n)\|_{2,\infty} \leqslant \|x\|_2 \leqslant C\, \|(a_n)\|_{2,1}, \tag{11}$$

where $\|\cdot\|_{2,\infty}$ and $\|\cdot\|_{2,1}$ are natural Lorentz sequence space norms. In particular such a basis is p-Besselian for each $p>2$.

Let us recall the definition of the relevant Lorentz norms. For a sequence $(a_n)_{n=1}^\infty$ we denote by $(a_n^*)_{n=1}^\infty$ the non-increasing rearrangement of the sequence $(|a_n|)_{n=1}^\infty$. Then $\|(a_n)_{n=1}^\infty\|_{2,\infty} = \sup_n \sqrt{n}\, a_n^*$ and $\|(a_n)_{n=1}^\infty\|_{2,1} = \sum_{n=1}^\infty n^{-1/2} a_n^*$.

Proof. First observe that applying Khintchine's inequality (or type and co-type 2 of the Hilbert space) we infer from Proposition 2 that for each finite set of indices A and each choice of signs we have $\|\sum_{n\in A} \pm x_n\| \sim \sqrt{|A|}$. Now let us denote $n_k = |\{n : |a_n| \geqslant 2^{-k}\}|$. Reordering the series $\sum_n a_n x_n$ so that $|a_n| \searrow 0$ we have

$$\begin{aligned}\left\|\sum_n a_n x_n\right\| &\leqslant 2\sum_k 2^{-k} \left\|\sum_{s=1}^{n_k} x_s\right\| \leqslant C\sum_k 2^{-k}\sqrt{n_k}\\ &\leqslant C\sum_{n=1} \frac{1}{\sqrt{n}}\,|a_n|.\end{aligned}$$

To prove the other inequality observe that from the Abel transform we obtain that if $\sum_{n=1}^\infty y_n$ converges in a Banach space X and $\sup_N \|\sum_{n=1}^N y_n\| = C$ and $\alpha_n \searrow 0$ then the series $\sum_{n=1}^\infty \alpha_n y_n$ converges and

$\sup_N \|\sum_{n=1}^{\infty} \alpha_n y_n\| \leqslant C\alpha_1$. Now we consider the series $\sum_{n=1}^{\infty} a_n x_n$ and assume that $|a_n| \searrow 0$. Since the basis is quasi-greedy this series converges and $\sup_N \|\sum_{n=1}^{N} a_n x_n\| \leqslant C \|x\|$, so for each N we have $\sup_k \|\sum_{s=1}^{k} a_{N+1-s} x_{N+1-s}\| \leqslant 2C \|x\|$. Applying our observation to the sequence $\alpha_k = |a_n| \, |a_{N+1-k}|^{-1}$ we get

$$\left\| \sum_{s=1}^{N} \frac{a_{N+1-s} \, |a_N|}{|a_{N+1-s}|} x_{N+1-s} \right\| \leqslant 2C \|x\|.$$

From this, using the unconditionality for constant coefficients we get

$$|a_N| \left\| \sum_{s=1}^{N} x_s \right\| \leqslant C \|x\|$$

which gives $\sup_n |a_n| \sqrt{n} \leqslant C \|x\|$ which completes the proof. ∎

3. OPTIMALITY

Suppose now that X is a quasi-Banach space with an unconditional basis (x_n, x_n^*) and let us assume that $\inf_{n \in F} \|x_n\| > 0$ so $\sup_{n \in F} \|x_n^*\| < \infty$. An unconditional basis is called a lattice basis if $\|\sum_n a_n x_n\| \leqslant \|\sum_n b_n x_n\|$ whenever $|a_n| \leqslant |b_n|$ for all n. If we have an unconditional basis we can always introduce an equivalent lattice norm by

$$|||x||| = \sup \left\{ \left\| \sum_n a_n x_n^*(x) \, x_n \right\| : |a_n| \leqslant 1 \right\}.$$

With this norm we have $\|x\| \leqslant |||x||| \leqslant C \|x\|$ and $|||x_n||| = \|x_n\|$.

Proposition 7. *Let X be a quasi-Banach space with lattice basis (x_n, x_n^*). For each x and each $m = 1, 2, \ldots$ there exists an element $T_m(x)$ of best m-term approximation i.e. $T_m(x) = \sum_{n \in A} a_n x_n$ with $|A| = m$ and $\|x - T_m(x)\| = \sigma_m(x)$.*

Proof. Let $x_k = \sum_{n \in A_k} a_n^k x_n$ with $|A_k| = m$ be such that $\|x - x_k\| \to \sigma_m(x)$. Using a standard diagonal procedure we can assume that for each n $\lim_{k \to \infty} a_n^k = a_n$. Clearly the a_n are not zero for at most m indices n. Write $x^\infty = \sum_n a_n x_n = \sum_{n \in A} a_n x_n$ where $|A| = m$. If we take B a finite set, $B \supset A$, then

$$\|x - x_k\| \geqslant \|P_B x - P_B x_k\| \to \|P_B x - x^\infty\|.$$

Thus for each such set B we have $\sigma_m(x) \geqslant \|P_B x - x^\infty\| = \|P_B(x - x^\infty)\|$. Taking a sequence of B's exhausting the whole index set we obtain $\sigma_m(x) \geqslant \|x - x^\infty\|$, so we can put $T_m(x) = x^\infty$. ∎

Let us recall the following quantities essentially considered in [10]:

$$e_m = \sup_{x \in X} \frac{\|x - \mathcal{G}_m(x)\|}{\sigma_m(x)} \qquad \left(\text{with } \frac{0}{0} = 1\right)$$

$$\mu_m = \sup_{k \leqslant m} \frac{\sup\{\|\sum_{n \in A} x_n\| : |A| = k\}}{\inf\{\|\sum_{n \in A} x_n\| : |A| = k\}}.$$

The importance of those quantities is clear. The sequence e_m estimates the error between the greedy algorithm $\mathcal{G}_m$ and the best possible m-term approximation. The quantity μ_m measures some sort of asymmetry of the basis. The important fact is that they are closely connected.

Theorem 4. *Let (x_n, x_n^*) be a lattice basis in a quasi-Banach space X. Then for each $m = 1, 2, \ldots$ we have*

$$\frac{1}{2\alpha} \mu_m \leqslant e_m \leqslant 2\alpha \mu_m. \tag{12}$$

The proof of this theorem follows the ideas from [10].

Proof. Let us fix m and $x = \sum_n a_n x_n \in X$. Let $T_m(x) = \sum_{n \in A} b_n x_n$ be the best m-term approximation. Let $\mathcal{G}_m(x) = \sum_{n \in B} a_n x_n$. Note that

$$\begin{aligned} \|x - P_A x\| &= \|x - T_m(x) + P_A T_m(x) - P_A x\| = \|(Id - P_A)(x - T_m(x))\| \\ &\leqslant \|x - T_m(x)\| = \sigma_m(x). \end{aligned} \tag{13}$$

Thus we can take $T_m(x) = P_A(x)$. In order to estimate $\|x - \mathcal{G}_m(x)\|$ write

$$\begin{aligned} x - \mathcal{G}_m(x) &= x - P_A x + P_A x - P_B x = (x - P_A x) + P_{A \setminus B} x - P_{B \setminus A} x \\ &= P_{F \setminus B}(x - P_A x) + P_{A \setminus B} x \end{aligned}$$

so $\|x - \mathcal{G}_m(x)\| \leqslant \alpha(\|x - T_m(x)\| + \|P_{A \setminus B} x\|) \leqslant \alpha(\sigma_m(x) + \|P_{A \setminus B} x\|)$. Note now that $\max\{|x_n^*(x)| : n \in A \setminus B\} := c \leqslant \min\{|x_n^*(x)| : n \in B \setminus A\}$ and also $|A \setminus B| = |B \setminus A| \leqslant m$. This implies that $\|P_{A \setminus B} x\| \leqslant c \, \|\sum_{n \in A \setminus B} x_n\|$ and $\|P_{B \setminus A} x\| \geqslant c \, \|\sum_{n \in B \setminus A} x_n\|$. Thus estimating c from the second inequality and substituting it into the first we get

$$\|P_{A \setminus B} x\| \leqslant \frac{\|P_{B \setminus A} x\|}{\|\sum_{n \in B \setminus A} x_n\|} \cdot \|P_{A \setminus B} x\| \leqslant \mu_m \, \|P_{B \setminus A} x\| \leqslant \mu_m \sigma_m(x) \tag{14}$$

so we get

$$\|x-\mathcal{G}_m(x)\| \leqslant \alpha\sigma_m(x)(1+\mu_m) \leqslant 2\alpha\mu_m\sigma_m(x).$$

In order to prove the other inequality we will need the following

LEMMA 8. *For each m there exist disjoint sets A and B with $|A|=|B|\leqslant m$ such that* $\|\sum_{n\in A} x_n\| \, \|\sum_{n\in B} x_n\|^{-1} \geqslant (2\alpha)^{-1}\mu_m$.

Proof. If $\mu_m \leqslant 2\alpha$ the claim is obvious. Otherwise take sets A and B with $|A|=|B|\leqslant m$ such that $\|\sum_{n\in A} x_n\| \, \|\sum_{n\in B} x_n\|^{-1} > \max(2\alpha, \mu_m-\varepsilon)$. For simplicity write

$$a=\left\|\sum_{n\in A} x_n\right\| \qquad b=\left\|\sum_{n\in B} x_n\right\|$$

$$a_1=\left\|\sum_{n\in A\cap B} x_n\right\| \qquad a_2=\left\|\sum_{n\in A\setminus B} x_n\right\|.$$

With this notation we have $2<(1/\alpha)(a/b)\leqslant(1/\alpha)(a/a_1)$ so $a_1<(1/2\alpha)\,a$. This implies

$$\frac{a}{b} \leqslant \frac{\alpha(a_1+a_2)}{b} = \alpha\,\frac{a_1}{b} + \alpha\,\frac{a_2}{b} < \frac{a}{2b} + \alpha\,\frac{a_2}{b}$$

so $a_2/b > (1/2\alpha)(a/b)$. Thus it suffices to replace A by any set of proper cardinality which contains $A\setminus B$ and is disjoint with B. ∎

Now let us take sets as in Lemma 8 and denote $|A|=|B|=k\leqslant m$. Let $C\supset A$ be a set of cardinality m disjoint with B. Consider

$$x := (1+\varepsilon)\sum_{n\in B} x_n + (1+\varepsilon/2)\sum_{n\in C\setminus A} x_n + \sum_{n\in A} x_n. \tag{15}$$

Then $\mathcal{G}_m(x) = x - \sum_{n\in A} x_n$ so $\|x-\mathcal{G}_m(x)\| = \|\sum_{n\in A} x_n\|$. From (13) we see that

$$\sigma_m(x) = \min\{\|P_S x\| : S\subset B\cup C \text{ and } |S|=k\}$$
$$\leqslant \|P_B x\| \leqslant (1+\varepsilon)\left\|\sum_{n\in B} x_n\right\|.$$

This and Lemma 8 give

$$e_m \geqslant \frac{\|\sum_{n\in A} x_n\|}{\sigma_m(x)} \geqslant \frac{\|\sum_{n\in A} x_n\|}{(1+\varepsilon)\,\|\sum_{n\in B} x_n\|} \geqslant \frac{1}{(1+\varepsilon)\,2\alpha}\,\mu_m.$$

Since ε was arbitrary we get the claim. ∎

Remark. Actually one can show that for x defined in (15) we have $\sigma_m(x) \sim \|\sum_{n \in B} x_n\|$. Namely $\|P_S x\| \geqslant \frac{1}{1+\varepsilon} \|\sum_{n \in S} x_n\|$ and since $\mu_m \geqslant \|\sum_{n \in A} x_n\| \, \|\sum_{n \in S} x_n\|^{-1}$ from Lemma 8 we get

$$\left\| \sum_{n \in S} x_n \right\| \geqslant \mu_m^{-1} \left\| \sum_{n \in A} x_n \right\| \geqslant \frac{1}{2\alpha} \frac{\|\sum_{n \in B} x_n\|}{\|\sum_{n \in A} x_n\|} \left\| \sum_{n \in A} x_n \right\| \geqslant \frac{1}{2\alpha} \left\| \sum_{n \in B} x_n \right\|$$

so $\sigma_m(x) \geqslant \frac{1}{2(1+\varepsilon)\,\alpha} \|\sum_{n \in B} x_n\|$.

Remark. Observe that we need to have μ_m defined as $\sup_{k \leqslant m}$ in order to have the above estimate. As an example take $\ell_\infty^n \oplus \ell_1$ with the natural basis. Let $(f_j)_{j=1}^n$ be the basis in ℓ_∞^n and $(e_k)_{k=1}^\infty$ the basis in ℓ_1. For $m > n$ and $x := 2 \sum_{j=1}^n f_j + \sum_{k=1}^m e_k$ we have $\sigma_m(x) = 2$ and $\mathcal{G}_m(x) = 2 \sum_{j=1}^n f_j + \sum_{k=1}^{m-n} e_k$ so $\|x - \mathcal{G}_m(x)\| = n$ which gives $e_m \geqslant n$. Also $\mu_m = n$. But

$$\xi_m := \frac{\sup\{\|\sum_{n \in A} x_n\| : |A| = m\}}{\inf\{\|\sum_{n \in A} x_n\| : |A| = m\}} = \frac{m}{m-n}$$

so no estimate of the form $e_m \leqslant C\xi_m$ is valid for all m unless $C \geqslant n$. But n can be arbitrary.

For general biorthogonal systems we have the following result.

THEOREM 5. *Suppose $(x_n, x_n^*)_{n \in F}$ is a complete biorthogonal system in a quasi-Banach space X with $\|x_n\| = 1$ for $n \in F$. Assume that for some $0 < c \leqslant C$ and $0 < p \leqslant q \leqslant \infty$ we have*

$$c \left(\sum_{n \in F} |x_n^*(x)|^q \right)^{1/q} \leqslant \|x\| \leqslant C \left(\sum_{n \in F} |x_n^*(x)|^p \right)^{1/p}. \tag{16}$$

Then $e_m \leqslant K m^{1/p - 1/q}$ where K depends only on α, C and c.

The proof is similar to the proof of Theorem 4 and Theorem 2.1 from [11].

Proof. Let us fix an $x \in X$ and $m = 1, 2, \ldots$. For any given $\varepsilon > 0$ we fix almost best m-term approximation i.e. $T_m = \sum_{n \in A} b_n x_n$ such that $\|x - T_m\| \leqslant \sigma_m(x) + \varepsilon$. First note that for any finite subset $V \subset F$ we have

$$\begin{aligned} \|P_V x\| &\leqslant C \left(\sum_{n \in V} |x_n^*(p)|^p \right)^{1/p} \leqslant C \, |V|^{1/p - 1/q} \left(\sum_{n \in V} |x_n^*(x)|^q \right)^{1/q} \\ &\leqslant \frac{C}{c} \, |V|^{1/p - 1/q} \, \|x\|. \end{aligned} \tag{17}$$

Let $\mathcal{G}_m(x)=\sum_{n\in B} a_n x_n$. We write

$$\|x-\mathcal{G}_m(x)\|=\|x-P_A x+P_A x-\mathcal{G}_m(x)\| \tag{18}$$

$$\leqslant \alpha(\|x-P_A x\|+\|P_A x-\mathcal{G}_m(x)\|). \tag{19}$$

The first summand is estimated as

$$\|x-P_A x\| \leqslant \alpha(\|x-T_m\|+\|P_A(x-T_m)\|)$$

$$\leqslant \alpha(\sigma_m(x)+\varepsilon+\|P_A\|(\sigma_m(x)+\varepsilon))$$

so

$$\|x-P_A x\| \leqslant \alpha\left(\sigma_m(x)+\varepsilon+\frac{C}{c}|A|^{1/p-1/q}(\sigma_m(x)+\varepsilon)\right). \tag{20}$$

The second summand we write as

$$\|P_A(x)-P_B x\| \leqslant \alpha(\|P_{A\backslash B}x\|+\|P_{B\backslash A}x\|)$$

and obtain

$$\|P_{B\backslash A}x\|=\|P_{B\backslash A}(x-T_m)\| \leqslant \frac{C}{c}|B\backslash A|^{1/p-1/q}(\sigma_m(x)+\varepsilon). \tag{21}$$

To estimate the other summand we note that $|B\backslash A|=|A\backslash B|$ and $|x_n^*(x)| \geqslant |x_s^*(x)|$ whenever $n\in B\backslash A$ and $s\in A\backslash B$. Thus

$$\begin{aligned}
\|P_{A\backslash B}x\| &\leqslant C\left(\sum_{n\in A\backslash B}|x_n^*(x)|^p\right)^{1/p} \leqslant \left(\sum_{n\in B\backslash A}|x_n^*(x)|^p\right)^{1/p}\\
&\leqslant C\,|B\backslash A|^{1/p-1/q}\left(\sum_{n\in B\backslash A}|x_n^*(x)|^q\right)^{1/q}\\
&= C\,|B\backslash A|^{1/p-1/q}\left(\sum_{n\in B\backslash A}|x_n^*(x-T_m)|^q\right)^{1/q}\\
&\leqslant \frac{C}{c}|B\backslash A|^{1/p-1/q}\|x-T_m\|\\
&\leqslant \frac{C}{c}|B\backslash A|^{1/p-1/q}(\sigma_m(x)+\varepsilon).
\end{aligned} \tag{22}$$

Since ε was arbitrary and $|B\backslash A| \leqslant m=|A|$ from (20), (21) and (22) we obtain $\|x-\mathcal{G}_m(x)\| \leqslant K(C,c,\alpha)\,\sigma_m(x)\cdot m^{1/p-1/q}$. ∎

Remark. Using Theorem 3 and arguing like in the above proof we can get that for each quasi-greedy basis in a Hilbert space we have $e_m \leqslant C\ln(m+1)$.

Now we will list some immediate consequence of Theorem 5.

COROLLARIES.

(a) *If $(x_n)_{n=1}^{\infty}$ is a complete, uniformly bounded orthonormal system (in particular the trigonometric system) then in $L_p[0,1]$ with $1 \leqslant p \leqslant \infty$ we have $e_m \leqslant Km^{|1/2-1/p|}$. This follows immediately from F. Riesz inequality which says that for $2 \leqslant p \leqslant \infty$ we have*

$$\left(\sum_{n=1}^{\infty} |\langle x_n, f\rangle|^2\right)^{1/2} \leqslant \|f\|_p \leqslant M\left(\sum_{n=1}^{\infty} |\langle x_n, f\rangle|^{p'}\right)^{1/p'}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, and also the dual inequality valid for $1 \leqslant p \leqslant 2$. This proves Theorem 2.1 from [11]. This is an optimal inequality as was shown for the trigonometric system in [11] Remark 2. For $p>1$ it also follows from Remark 2.

(b) *Since for any semi-normalized biorthogonal system (x_n, x_n^*) in a Banach space X we have*

$$c\sup_n |x_n^*(x)| \leqslant \|x\| \leqslant C\sum_n |x_n^*(x)|$$

we infer that for each such system we have $e_m \leqslant Cm$. This estimate is also optimal, even for unconditional bases. One easily checks that for the natural unconditional basis in $\ell_1 \oplus c_0$ one has $e_m \geqslant m$.

(c) *In each super-reflexive space, in particular in $L_p[0,1]$ with $1<p<\infty$, each semi-normalized basis (x_n, x_n^*) satisfies equation (16) for some $1<q\leqslant p<\infty$ (see [1]). So we obtain that for each semi-normalized basis in a super-reflexive space we have $e_m \leqslant Km^{\beta}$ with $\beta<1$.*

(d) *If (x_n, x_n^*) is a semi-normalized unconditional basis in L_p with $1<p<\infty$, then for $p \geqslant 2$ it satisfies*

$$c\left(\sum_{n\in F} |x_n^*(x)|^p\right)^{1/p} \leqslant \|x\|_p \leqslant C\left(\sum_{n\in F} |x_n^*(x)|^2\right)^{1/2}$$

and the dual inequality for $1<p\leqslant 2$. Thus for an unconditional basis in L_p we have $e_m \leqslant Km^{|1/2-1/p|}$. Also this estimate is optimal. To see it consider L_p as being isomorphic to $\ell_2 \oplus L_p$ and take the natural basis in ℓ_2 and the Haar basis in L_p.

4. MULTIPLE HAAR SYSTEM

In this section we will discuss the efficiency of the greedy algorithm with respect to the multi-dimensional Haar wavelet. For a more detailed exposition of the general background sketched below the reader may consult [13]. We will argue in the context of the square function for $0<p<\infty$. To start we define

$$H(t)=\begin{cases}1 & \text{if} \quad t\in[0,\frac{1}{2}),\\ -1 & \text{if} \quad t\in[\frac{1}{2},1),\\ 0 & \text{otherwise.}\end{cases} \tag{23}$$

For a dyadic interval $I=[k2^{-n},(k+1)\,2^{-n})$ we put $h_I(t)=2^{n/p}H(2^nt-k)$. For a dyadic rectangle in $J=I_1\times\cdots\times I_d\subset\mathbb{R}^d$ we put

$$h_J^{(d)}(t)=h_{I_1}(t_1)\cdots h_{I_d}(t_d). \tag{24}$$

The set of all dyadic intervals in $\mathbb{R}$ will be denoted by $\mathscr{D}(1)$ and the set of all dyadic rectangles in $\mathbb{R}^d$ will be denoted by $\mathscr{D}(d)$. The system $(h_i^{(d)})_{I\in\mathscr{D}(d)}$ is a complete orthogonal system in $L_2(\mathbb{R}^d)$ and is normalized in $L_p(\mathbb{R}^d)$. Note that formally the definition of this system depends on p but since p will be fixed in our future arguments we will not indicate this dependence explicitely.

A function $f=\sum_{I\in\mathscr{D}(d)}a_Ih_I^{(d)}$ is in $H_p(\mathbb{R}^d)$ if the norm

$$|||f|||=\left(\int_{\mathbb{R}^d}\left(\sum_{I\in\mathscr{D}(d)}|a_Ih_I^{(d)}(t)|^2\right)^{p/2}dt\right)^{1/p} \tag{25}$$

is finite. It is known by the Littlewood–Paley theory that for $1<p<\infty$ this norm is equivalent to the usual L_p norm and we have $H_p(\mathbb{R}^d)=L_p(\mathbb{R}^d)$. For $0<p\leqslant 1$ we get the dyadic H_p-space.

The main result of this section is the following

THEOREM 6. *For $0<p<\infty$ and $d=1,2,...$ for the system $(h_I^{(d)})_{I\in\mathscr{D}(d)}$ in $H_p(\mathbb{R}^d)$ we have*

$$e_m\sim(\log m)^{(d-1)\,|1/2-1/p|}. \tag{26}$$

This result substantiates the conjecture formulated (for $p>1$) in [10] and extends results from [9] and [10]. Our argument is a modification of the argument from [2]. Let us start with a lemma which summarises the argument from the first few lines of the proof of Proposition 3.3 from [2]. We repeat the short proof of this lemma for the convenience of the reader.

LEMMA 9. *For $0<p<\infty$ and any finite subset $B\subset\mathscr{D}(1)$ we have*

$$2^{-1/p}\,|B|^{1/p}\leqslant\left|\!\left|\!\left|\sum_{I\in B}h_I\right|\!\right|\!\right|.$$

Proof. Let us denote $2^{M(t)}=\max_{I\in B}|h_I(t)|^p$. From the definition of the Haar system we infer that $2^{M(t)}\geqslant\frac{1}{2}\sum_{I\in B}|h_I(t)|^p$ so

$$\left|\!\left|\!\left|\sum_{I\in B}h_I\right|\!\right|\!\right|\geqslant\left(\int_{\mathbb{R}}2^{M(t)}\,dt\right)^{1/p}\geqslant\left(\frac{1}{2}\int_{\mathbb{R}}\sum_{I\in B}|h_I(t)|^p\,dt\right)^{1/p}=2^{-1/p}\,|B|^{1/p}. \quad\blacksquare$$

PROPOSITION 10. *For any $d=1,2,\ldots$, any finite $B\subset\mathscr{D}(d)$, $|B|=m$ and any numbers $(a_I)_{I\in B}$ we have*

(a) *if $0<p\leqslant 2$ then*

$$(\log m)^{(1/2-1/p)\,d}\left(\sum_{I\in B}|a_I|^p\right)^{1/p}\leqslant\left|\!\left|\!\left|\sum_{I\in B}a_Ih_I^{(d)}\right|\!\right|\!\right|\leqslant\left(\sum_{I\in B}|a_I|^p\right)^{1/p}\tag{27}$$

(b) *if $2\leqslant p<\infty$ then*

$$\left(\sum_{I\in B}|a_I|^p\right)^{1/p}\leqslant\left|\!\left|\!\left|\sum_{I\in B}a_Ih_I^{(d)}\right|\!\right|\!\right|\leqslant(\log m)^{(1/2-1/p)\,d}\left(\sum_{I\in B}|a_I|^p\right)^{1/p}.\tag{28}$$

Proof. The right hand side inequality in (27) is easy (it is actually the type of H_p). We simply apply the Hölder's inequality with exponent $\frac{2}{p}\geqslant 1$ to the inside sum and we get

$$\left(\int_{\mathbb{R}^d}\left(\sum_{I\in B}|a_Ih_I^{(d)}(t)|^2\right)^{p/2}dt\right)^{1/p}\leqslant\left(\int_{\mathbb{R}^d}\sum_{I\in B}|a_Ih_I^{(d)}(t)|^p\,dt\right)^{1/p}$$
$$=\left(\sum_{I\in B}|a_I|^p\right)^{1/p}.$$

Now let $d=1$ and $0<p\leqslant 2$. Let σ: $\{1,2,\ldots,|B|\}\to B$ be such that $|a_{\sigma(i)}|$ is a decreasing sequence. Fix s such that $2^{s-1}<m\leqslant 2^s$ and put $f_k=(\sum_{j=2^{k-1}+1}^{2^k}|a_{\sigma(j)}h_{\sigma(j)}|^2)^{1/2}$. Then

$$\left|\!\left|\!\left|\sum_{I\in B}a_Ih_I\right|\!\right|\!\right|=\left(\int_{\mathbb{R}}\left(\sum_{k=0}^{s}f_k^2(t)\right)^{p/2}dt\right)^{1/p}=\left(\int_{\mathbb{R}}\left(\sum_{k=0}^{s}(f_k^p(t))^{2/p}\right)^{p/2}dt\right)^{1/p}$$
$$\geqslant\left(\left(\sum_{k=0}^{s}\left(\int_{\mathbb{R}}f_k^p(t)\,dt\right)^{2/p}\right)^{p/2}\right)^{1/p}$$
$$=\left(\sum_{k=0}^{s}\left|\!\left|\!\left|\sum_{j=2^{k-1}+1}^{2^k}a_{\sigma(j)}h_{\sigma(j)}\right|\!\right|\!\right|^2\right)^{1/2}$$

$$\geqslant \left(\sum_{k=0}^{s} \left\| \sum_{j=2^{k-1}+1}^{2^k} a_{\sigma(2^k)} h_{\sigma(j)} \right\|^2 \right)^{1/2}$$

and from Lemma 9

$$\geqslant \left(\sum_{k=0}^{s} 2^{2(k-1)/p} \, |a_{\sigma(2^k)}|^2 \right)^{1/2}.$$

Since

$$\sum_{I \in B} |a_I|^p = \sum_{j=1}^{|B|} |a_{\sigma(j)}|^p \leqslant \sum_{k=0}^{s} 2^k \, |a_{\sigma(2^k)}|^p \leqslant s^{1-p/2} \left(\sum_{k=0}^{s} 2^{2k/p} \, |a_{\sigma(2^k)}|^2 \right)^{p/2}$$

we get

$$\left\| \sum_{I \in B} a_I h_I \right\| \geqslant 2^{-1/p} (\log m)^{-(1-p/2)\,1/p} \left(\sum_{I \in B} |a_I|^p \right)^{1/p}$$

$$= 2^{-1/p} (\log m)^{1/2-1/p} \left(\sum_{I \in B} |a_I|^p \right)^{1/p}.$$

Now we will prove the left hand side inequality in (27) by induction on d. Suppose we have (27) valid for $d-1$. Given a finite set $B \subset \mathscr{D}(d)$ we write each $I \in B$ as $I = J \times K$ with $J \in \mathscr{D}(1)$ and $K \in \mathscr{D}(d-1)$ and then $h_I^{(d)}(t) = h_J(t_1) \cdot h_K^{(d-1)}(\xi)$ where $\xi = (t_2, ..., t_d)$. Now we estimate

$$\left\| \sum_{I \in B} a_I h_I^{(d)} \right\| = \left(\int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \left(\sum_{I \in B} |a_I h_J(t_1)|^2 \, |h_K^{(d-1)}(\xi)|^2 \right)^{p/2} dt_1 \, d\xi \right)^{1/p}$$

$$= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} \left(\sum_K \left(\sum_J |a_I h_J(t_1)|^2 \right) |h_K^{(d-1)}(\xi)|^2 \right)^{p/2} d\xi \right) dt_1 \right)^{1/p}. \tag{29}$$

For each t_1 we apply the inductive hypothesis (note that the number of different K's is at most $|B|$) and we continue the estimates

$$\geqslant C(d-1, p) (\log |B|)^{(d-1)(1/2-1/p)} \times \left(\int_{\mathbb{R}} \sum_K \left(\sum_J |a_I h_J(t_1)|^2 \right)^{p/2} dt_1 \right)^{1/p}$$

$$\geqslant C(d-1, p) (\log |B|)^{(d-1)(1/2-1/p)} \times \left(\sum_K \int_{\mathbb{R}} \left(\sum_J |a_I h_J(t_1)|^2 \right)^{p/2} dt_1 \right)^{1/p}. \tag{30}$$

Now we apply the estimate (27) for $d=1$ and we continue as

$$\geqslant C(d-1,p)(\log|B|)^{(d-1)(1/2-1/p)}\left(\sum_K\sum_J|a_I|^p\right)^{1/p} C(1,p)(\log|B|)^{(1/2-1/p)}$$

$$=C(d,p)(\log|B|)^{d(1/2-1/p)}\left(\sum_{I\in B}|a_I|^p\right)^{1/p}. \tag{31}$$

The inequality (28) follows by duality from (27) for $1<p\leqslant 2$. ∎

PROPOSITION 11. *For every finite set $B\subset\mathscr{D}(d)$ we have*

(a) *if $0<p\leqslant 2$ then*

$$C(d,p)(\log|B|)^{(1/2-1/p)(d-1)}|B|^{1/p}\leqslant\left|\!\left|\!\left|\sum_{I\in B}h_I^{(d)}\right|\!\right|\!\right|\leqslant|B|^{1/p} \tag{32}$$

(b) *if $2\leqslant p<\infty$ then*

$$|B|^{1/p}\leqslant\left|\!\left|\!\left|\sum_{I\in B}h_I^{(d)}\right|\!\right|\!\right|\leqslant C(d,p)(\log|B|)^{(1/2-1/p)(d-1)}|B|^{1/p}. \tag{33}$$

Proof. As in the previous Proposition 10 inequality (33) follows by duality from (32). Note also that (32) for $d=1$ is Lemma 9. For $d>1$ we proceed like in the proof of Proposition 10. We write each $I\in B$ as $J\times K$ and estimate $|\!|\!|\sum_{I\in B}h_i^{(d)}|\!|\!|$ exactly like in (29) and (30). Since $a_I=1$ instead of (27) for $d=1$ we apply Lemma 9 and we obtain

$$\left|\!\left|\!\left|\sum_{I\in B}h_I^{(d)}\right|\!\right|\!\right|\geqslant C(d-1,p)(\log|B|)^{(1/2-1/p)(d-1)}2^{-1/p}|B|^{1/p}.\ ∎$$

Proof of Theorem 6. The estimate $e_m\leqslant C(\log m)^{|1/2-1/p|(d-1)}$ follows immediately from Theorem 4 and Proposition 11. The estimate from below was proved in [9]. ∎

Theorem 6 covers and extends the main results about the Haar system proved in [9] and [10]. In particular it gives a new proof that the Haar wavelet is a greedy basis in $L_p(\mathbb{R})$. One can note that the Haar system is not the only such basis in L_p for $2<p<\infty$. Let us recall the definition of the Rosenthal space (cf. [6] p. 169). Fix $0<\beta\leqslant 1$ and define a norm on sequences $(a_n)_{n=1}^\infty$ as

$$\|(a_n)\|^\beta=\left(\sum_{n=1}^\infty|a_n|^p\right)^{1/p}+\left(\sum_{n=1}^\infty|a_nw_n^\beta|^2\right)^{1/2}$$

with $w_n^\beta = n^{-\beta(p-2)/2p}$. It is known that for all β's we get the same space X (called Rosenthal space) and that $X \oplus L_p$ is isomorphic to L_p. However for different β's we get different unconditional bases in X. If $(e_n)_{n=1}^\infty$ denotes the unit vector basis then for each finite set $A \subset \mathbb{N}$ we have

$$\begin{aligned}\Big\| \sum_{n \in A} e_n \Big\|^\beta &= |A|^{1/p} + \Big(\sum_{n \in A} |w_n^\beta|^2 \Big)^{1/2} \\ &\leqslant |A|^{1/p} + \Big(\sum_{n=1}^{|A|} |w_n^\beta|^2 \Big)^{1/2} \\ &\leqslant |A|^{1/p} + C\,|A|^{(1-\beta)/2 - \beta/p}\end{aligned}$$

For $\beta = 1$ we get $\|\sum_{n \in A} e_n\|^1 \sim |A|^{1/p}$ so this basis and the Haar basis give an unconditional greedy basis in L_p which is not equivalent to the Haar basis.

For $0 < \beta < 1$ we get $\|\sum_{n=1}^N e_n\|^\beta \sim N^{(1-\beta)/2 + \beta/p}$ so for this basis in X and the Haar basis in L_p we get an unconditional basis in L_p with

$$\mu_m \sim m^{(1-\beta)/2 + \beta/p} m^{-1/p} \sim m^{(1-\beta)(1/2 - 1/p)}$$

so we get all possible power type behaviours of e_m (c.f. Corollary (d) after Theorem 5).

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